Probability And Random Variables

(1) Find the value of k for a cts rv x whose density func. is given by $f(x)=kx^2e^{-x}, x\geq 0$.

Ans: WKT $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$

$u=x^2 \qquad dv=e^{-x}dx$

$u'=2x \qquad v=\frac{e^{-x}}{-1}$

$u''=2 \qquad v_1=\frac{e^{-x}}{(-1)^2}$

$v_2=\frac{e^{-x}}{(-1)^3}$

$$k\int_0^{\infty} x^2e^{-x}dx = 1$$

$$k\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)_0^{\infty} = 1$$

$$2k = 1 \Rightarrow k = 1/2$$

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(2) Let A & B be two events such that $P(A)=0.5$, $P(B)=0.3$ & $P(A\cap B)=0.15$ compute $P(B|A)$ & $P(\bar{A}\cap B)$.

Ans: Given $P(A)=0.5$, $P(B)=0.3$, $P(A\cap B)=0.15$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.15}{0.5} = 0.3$$

$$P(\bar{A}\cap B) = P(B) - P(A\cap B) = 0.3 - 0.15 = 0.15$$

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(3) Let A & B be 2 events such that $P(A)=1/3$, $P(B)=3/4$ & $P(A\cap B)=1/4$. Compute $P(A|B)$ & $P(\bar{A}\cap\bar{B})$.

Ans: $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/4}{3/4} = 1/3$

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B) = 1 - [P(A)+P(B)-P(A\cap B)]$$

$$= 1 - [1/3 + 3/4 - 1/4] = 1/6$$

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(4) A bag contains 8 white & 4 black balls. If 5 balls are drawn at random, what is the probability that 3 are white & 2 are black?

Ans: Total no. of balls $= 8+4 = 12$

$S = \{5 \text{ balls are taken out of } 12\}$

$$n(S) = 12C_5 = \frac{12\times11\times10\times9\times8}{1\times2\times3\times4\times5} = 792$$

$n(A) = 8C_3 \times 4C_2 = 336$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{336}{792} = 0.4242$$

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(5) Let $M_x(t) = \frac{1}{1-t}$, $|t|<1$, the mgf of rv x. Find $E(x)$ & $E(x^2)$.

Ans: Given $M_x(t) = \frac{1}{1-t} = (1-t)^{-1} = 1+t+t^2+\cdots = 1+t+2\cdot\frac{t^2}{2!}+\cdots$

$E(x)$ = coeff. of t in $M_x(t) = 1$

$E(x^2)$ = " " $\frac{t^2}{2!}$ " " $= 2$

(6) A rv x has probability mass funf. $P(x=x) = \frac{x}{10}$, $x = 1,2,3,4$. Find the cumulative distribution funf. $F(x)$ of x.

Ans:

x :	1	2	3	4
$P(x)$:	$1/10$	$2/10$	$3/10$	$4/10$

x	$P(x)$	$F(x) = P(x \le x)$
1	$1/10$	$F(1) = P(1) = 1/10$
2	$2/10$	$F(2) = P(1)+P(2) = 1/10 + 2/10 = 3/10$
3	$3/10$	$F(3) = P(1)+P(2)+P(3) = 6/10$
4	$4/10$	$F(4) = P(1)+P(2)+P(3)+P(4) = 1$

(7) The rv x has pmf $P(x=x) = \begin{cases} c/x, & x=1,2,3 \\ 0, & \text{otherwise} \end{cases}$. Obtain (i) the value of c (ii) $P(x \ge 2)$.

Ans:

x :	1	2	3
$P(x)$:	$c/1$	$c/2$	$c/3$

(i) $\sum_{x=1}^{3} P(x) = 1$

$\frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1$

$c\left(\frac{11}{6}\right) = 1$

$c = \frac{6}{11}$

(ii) $P(x \ge 2) = P(2) + P(3)$

$= c/2 + c/3 = \frac{5c}{6} = \frac{5}{6} \times \frac{6}{11} = \frac{5}{11}$

(8) S.T. for any events A & B in S, $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

Ans: $RHS = P(B/A)P(A) + P(B/\bar{A})P(\bar{A})$

$= \frac{P(A\cap B)}{P(A)} P(A) + \frac{P(\bar{A}\cap B)}{P(\bar{A})} P(\bar{A})$

$= P(A\cap B) + P(\bar{A}\cap B) = P(A\cap B) + P(B) - P(A\cap B) = P(B)$

$= LHS$

⑨ The pdf of the rv x is given by $f(x)=\begin{cases} k(1-x^2), & 0<x<1 \\ 0, & \text{otherwise}\end{cases}$. Find the value of k.

Ans: WKT $\int_{-\infty}^{\infty} f(x)dx = 1$

$$k\int_0^1 (1-x^2)\,dx = 1$$

$$k\left(x-\frac{x^3}{3}\right)_0^1 = 1 \Rightarrow k\left(1-\frac{1}{3}\right)=1 \Rightarrow k = \frac{3}{2}$$

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⑩ If A & B are mutually exclusive events $P(A)=0.29$ & $P(B)=0.43$ then find $P(\bar{A})$ & $P(A\cup B)$.

Ans: $P(\bar{A}) = 1-P(A) = 1-0.29 = 0.71$

$P(A\cup B) = P(A)+P(B) = 0.29+0.43 = 0.72$

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⑪ A coin is tossed 3 times. If x denotes the number of heads obtained, find the probability distribution of x.

Ans: Probability distribution of x: x – number of heads obtained

x:	0	1	2	3
$P(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

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⑫ State memory less property & which continuous & discrete distributions follow this property?

Ans: Memoryless property:

If x is a random variable, then $P(x>m+n / x>m) = P(x>n)$ for any $m,n>0$.

In continuous distribution, exponential distribution follows memoryless property & in discrete distribution, geometric distribution follows memoryless property.

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⑬ A random variable x has a uniform distribution over $(-3,3)$. Compute $P(|x-2|<2)$.

Ans: Given x is a uniform random variable in the interval $(-3,3)$.

(ie) $a=-3$, $b=3$.

Pdf: $f(x) = \frac{1}{b-a} = \frac{1}{3+3} = \frac{1}{6}$, $-3 < x < 3$

$$P(|x-2| < 2) = P(-2 < x-2 < 2) = P(-2+2 < x < 2+2)$$

$$= P(0 < x < 4) = \int_0^3 f(x)\,dx = \int_0^3 \tfrac{1}{6}\,dx$$

$$= \frac{1}{6}(x)_0^3 = \frac{1}{6}(3-0) = \tfrac{1}{2}$$

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(14) If the pdf of x is given by $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ then show that $E(x^r) = \dfrac{2}{(r+1)(r+2)}$.

Ans: Given $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^1 x^r\, 2(1-x)\,dx = 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1 = 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{r+2-r-1}{(r+1)(r+2)}\right] = 2\left[\frac{1}{(r+1)(r+2)}\right] = \frac{2}{(r+1)(r+2)}$$

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(15) Find k, if the pdf of x is

x :	-1	0	1	2	3
$P(x=x)$:	$2k$	$3k$	$4k$	$6k^2$	$4k^2$

Ans: WKT $\sum P(x) = 1$

$2k + 3k + 4k + 6k^2 + 4k^2 = 1$

$9k + 10k^2 = 1$

$10k^2 + 9k - 1 = 0$

$(10k-1)(k+1) = 0$

$\times \quad +$
$-10 \quad 9$
$-1 \quad +10$
$10k-1 \quad 10k+10$
$k+1$

$10k = 1$ | $k+1 = 0$

$k = 1/10$ | $k = -1$

$\therefore k = 1/10$

Two Dimensional Random Variables

① Let X & Y be two independent rvs with $Var(X)=9$ & $Var(Y)=3$. Find $Var(4X-2Y+6)$.

Sol: $Var(4X-2Y+6) = 4^2 Var(X) + (-2)^2 Var(Y) = (16\times 9) + (4\times 3) = 156$

② The joint pdf of (X,Y) is $f(x,y) = \begin{cases} 4xy, & 0\le x\le 1, 0\le y\le 1/2 \\ 0, & \text{otherwise} \end{cases}$. Calculate $P(X\le 2Y)$.

Sol: $$P(X\le 2Y) = \int_0^{1/2}\int_0^{2y} 4xy\,dx\,dy$$

$$= 4\int_0^{1/2} y\left(\frac{x^2}{2}\right)_0^{2y} dy = 2\int_0^{1/2} y(4y^2)\,dy = 8\left(\frac{y^4}{4}\right)_0^{1/2} = 2\left(1/16\right) = 1/8$$

③ Define covariance & coefficient of correlation between 2 rvs X & Y.

Sol: $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$\sigma_X = \sqrt{Var\,X}$ & $\sigma_Y = \sqrt{Var\,Y}$, $Var(X) = E(X^2) - [E(X)]^2$, $Var(Y) = E(Y^2) - [E(Y)]^2$

④ The joint pdf of a bivariate rv (X,Y) is given by $f(x,y) = \begin{cases} k, & 0<y\le x<1 \\ 0, & \text{otherwise} \end{cases}$ where k is a constant. Determine the value of k.

Sol: WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$k\int_0^1\int_0^x dy\,dx = 1 \Rightarrow k\int_0^1 (y)_0^x dx = 1 \Rightarrow k\int_0^1 x\,dx = 1$$

$$k\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow k\left(1/2\right) = 1 \Rightarrow k = 2$$

⑤ P.T. the correlation coeff. ρ_{xy} of the rvs X & Y takes value in the range -1 & 1.

Sol: WKT $r = \frac{\rho}{\sigma_X \sigma_Y}$

$$\rho^2 = \left[\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}\right]^2 = \left(\frac{\Sigma XY}{n}\right)^2$$

$$\sigma_X^2 \sigma_Y^2 = \frac{\Sigma X^2 \Sigma Y^2}{n^2}$$

$$(\Sigma (xy))^2 \le (\Sigma x^2)(\Sigma y^2)$$

$$\frac{(\Sigma xy)^2}{n^2} \le \frac{\Sigma x^2 \Sigma y^2}{n^2}$$

$$\rho^2 \le \sigma_x^2 \sigma_y^2$$

$$(r\sigma_x\sigma_y)^2 \le \sigma_x^2\sigma_y^2$$

$$r^2\sigma_x^2\sigma_y^2 \le \sigma_x^2\sigma_y^2$$

$$r^2 \le 1 \Rightarrow |r| \le 1 \Rightarrow -1 \le r \le 1$$

(6) Let (x,y) be a two-dimensional rv. Define covariance of (x,y). If x & y are independent. What will be the covariance of (x,y)?

Sol: $Cov(x,y) = E(xy) - E(x)E(y)$

$= E(x)E(y) - E(x)E(y)$ ($\because x$ & y are independent)

$= 0$

(7) Can $y = 5 + 2.8x$ & $x = 3 - 0.5y$ be the estimated regression eqn. of y on x respectively explain your answer.

Sol: $b_{yx} = 2.8$, $b_{xy} = -0.5$

$r = \pm\sqrt{b_{xy} \cdot b_{yx}}$ = imaginary

$\therefore$ They cannot be estimated regression eqns.

(8) The joint pmf of a 2-dimensional rv (x,y) is given by $P(x,y) = k(2x+y)$, $x = 1,2$ & $y = 1,2$, where k is a constant. Find the value of k.

Sol:

$x \backslash y$	1	2
1	3k	4k
2	5k	6k

$18k = 1$

$k = 1/18$

(9) If the joint pdf of (x,y) is $f(x,y) = \begin{cases} 1/4, & 0 \le x, y \le 2 \\ 0, & \text{otherwise} \end{cases}$, find $P(x+y \le 1)$.

Sol: $P(x+y \le 1) = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy = \frac{1}{4}\int_0^1\int_0^{1-y} dx\,dy$

$= \frac{1}{4}\int_0^1 (x)_0^{1-y} dy = \frac{1}{4}\int_0^1 (1-y)\,dy = \frac{1}{4}\left(y - \frac{y^2}{2}\right)_0^1$

$= \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$

(10) Determine the value of the constant c if the joint density func. of 2 discrete rvs X & Y is given by $p(m,n) = cmn$, $m = 1,2,3$ & $n = 1,2,3$.

Sol: Given $p(m,n) = cmn$, $m = 1,2,3$ & $n = 1,2,3$

n \ m	1	2	3
1	c	2c	3c
2	2c	4c	6c
3	3c	6c	9c

$36c = 1$

$c = 1/36$

(11) Determine the value of k if $f(x,y) = \begin{cases} kxe^{-y} & ; 0 < x < 2, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$ is a joint pdf of two dimensional rv (x, y).

Ans: Given $f(x,y) = \begin{cases} kxe^{-y} & ; 0 < x < 2, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\int_0^2\int_0^{\infty} kxe^{-y}\,dy\,dx = 1 \Rightarrow k\int_0^2 x\left(\frac{e^{-y}}{-1}\right)_0^{\infty} dx = 1$

$-k\int_0^2 x(0-1)\,dx = 1 \Rightarrow k\int_0^2 x\,dx = 1 \Rightarrow k\left(\frac{x^2}{2}\right)_0^2 = 1$

$\frac{k}{2}(4-0) = 1 \Rightarrow 2k = 1 \Rightarrow \boxed{k = 1/2}$

(12) When will the 2 regression lines be (a) at right angles, (b) coincident?

Ans: When $r = 0$, $\tan\theta = \infty \Rightarrow \theta = \pi/2$ & so the regression lines are perpendicular.

When $r = 1$ or -1, $\theta = 0$ & so the regression lines are coincide.

(13) The joint pdf of (x, y) is $f(x,y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$. Calculate $P(x \leq 2y)$.

Ans: Given $f(x,y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$$P(X \le 2Y) = 4\int_0^1 \int_0^{2y} xy \, dx\,dy = 4\int_0^1 \left(\frac{x^2}{2}\right)_0^{2y} y\,dy$$

$$= \frac{4}{2}\int_0^1 (4y^2)\,y\,dy = 2\int_0^1 4y^3\,dy = 8\left(\frac{y^4}{4}\right)_0^1$$

$$= 2$$

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(14) State the central limit theorem for independent & identically distributed random variables.

<u>Ans:</u> If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E[X_i] = \mu$ & $Var[X_i] = \sigma^2$, $i = 1, 2, \ldots, n$ & if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

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(15) If $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$, is the joint pdf of (X, Y), find $P(X + Y \le 1)$.

<u>Ans:</u> Given $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$

$$P(X + Y \le 1) = \int_0^1 \int_0^{1-y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^1 \left(\frac{e^{-x}}{-1}\right)_0^{1-y} e^{-y}\,dy = -\int_0^1 \left(e^{-(1-y)} - 1\right) e^{-y}\,dy$$

$$= -\int_0^1 \left(e^{-1} - e^{-y}\right) dy = -\left(y e^{-1} - \frac{e^{-y}}{-1}\right)_0^1$$

$$= -\left(e^{-1} + e^{-1} - 1\right) = -\left(2e^{-1} - 1\right) = 1 - 2e^{-1}$$

RANDOM PROCESSES

① State the 4 types of a stochastic processes.

Ans: 1) Continuous random process 2) Continuous random sequence
3) Discrete random process 4) Discrete random sequence

② Define a stationary process.

Ans: A random process $X(t)$ is said to be stationary if its mean & variance are constant. (ie) $E[X(t)] = \text{constant}$ & $Var[X(t)] = \text{constant}$.

③ Define WSS process.

Ans: A random process $X(t)$ is said to be wide sense stationary if its mean is constant & its autocorrelation depends only on time difference.
(ie) $E[X(t)] = \text{constant}$ & $R(t_1, t_2) =$ is a func. of $t_1 - t_2$.

④ State Chapman-Kolmogorow theorem.

Ans: If P is the tpm of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n. (ie) $[P_{ij}^{(n)}] = [P_{ij}]^n$.

⑤ Consider a Markov chain with 2 states & tpm $P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$. Find the steady state probabilities of the chain.

Sol: WKT $\pi P = \pi$

$$(\pi_1, \pi_2) \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2) \quad \& \quad \pi_1 + \pi_2 = 1$$

$$\frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_1 \;; \quad \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_2$$

$$\Rightarrow -\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 = 0$$

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 = \frac{1}{4}$$

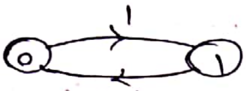
$$\frac{3}{4}\pi_2 = 1/4 \Rightarrow \pi_2 = 1/4 \times 4/3 = 1/3$$

$$\therefore \pi_1 = 2/3$$

$$\therefore \pi = (2/3 \;\; 1/3)$$

⑥ Consider a Markov chain with state $\{0, 1\}$ & tpm $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Is the state 0 periodic? If so what is the period?

Ans: $P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$



$d_i = GCD\{n \mid P_{ii}^{(n)} > 0\} = GCD\{2, 4, 6, \ldots\} = 2 = \text{Period}.$

State 0 is periodic with period 2.

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⑦ State any 2 properties of a Poisson process.

Ans: ① The Poisson process possess the Markov property.

② Sum of 2 independent Poisson processes is a Poisson process.

③ Difference of 2 independent Poisson processes is not a Poisson process.

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⑧ State the postulates of a Poisson process.

Ans: Let $X(t)$ = no. of times an event A say, occured upto time t so that the sequence $\{X(t), t \in [0, \infty)\}$ forms a Poisson process with parameter λ.

(i) Events occuring in non-overlapping intervals are independent of each other.

(ii) $P[X(t) = 1$ for t in $(x, x+h)] = \lambda h + o(h)$

(iii) $P[X(t) = 0$ for t in $(x, x+h)] = 1 - \lambda h + o(h)$

(iv) $P[X(t) = 2$ or more for t in $(x, x+h)] = o(h)$

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⑨ What is Markov process?

Ans: A random process $X(t)$ is said to be a Markov process if for any

$t_1 < t_2 < t_3 < \ldots < t_n$

$$P[X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_1) = x_1]$$

$$= P[X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}]$$

(ie) the conditional distribution of $X(t_n)$ for given values of $X(t_1), X(t_2), \ldots, X(t_{n-1})$ depends only on $X(t_{n-1})$.

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⑩ Let $X(t)$ be a WSS random process with $E[X(t)] = 0$ & $Y(t) = X(t) - X(t+\tau)$, $\tau > 0$. Compute $E[Y(t)]$ & $Var[Y(t)]$.

Ans: Given $Y(t) = X(t) - X(t+\tau)$

$$E[Y(t)] = E[X(t) - X(t+\tau)]$$

$$= E[X(t)] - E[X(t+\tau)]$$

$$= 0 - 0 = 0 \quad [\because E[X(t)] = E[X(t+\tau)], \ \because X(t) \text{ is WSS}]$$

$$Var[Y(t)] = E[Y^2(t)] - [E[Y(t)]]^2$$

$$= E[(x(t)-x(t+\tau))^2] - 0$$
$$= E[x^2(t)+x^2(t+\tau)-2x(t)x(t+\tau)]$$
$$= E[x^2(t)]+E[x^2(t+\tau)]-2E[x(t)x(t+\tau)]$$
$$= R_{xx}(0)+R_{xx}(0)-2R_{xx}(\tau)$$
$$= 2[R_{xx}(0)-R_{xx}(\tau)]$$

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(11) Let $\{X_n; n\geq 0\}$ be a Markov chain having state space $S=\{1,2\}$ & one-step tpm $P=\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$. Find the stationary probabilities of the Markov chain.

Ans: Given $P=\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$

WKT $\pi P = \pi$ & $\pi_1+\pi_2=1$ where $\pi=(\pi_1 \ \pi_2)$

$$(\pi_1 \ \pi_2)\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} = (\pi_1 \ \pi_2)$$

$\frac{1}{2}\pi_1 = \pi_1 \Rightarrow \pi_1 - \frac{1}{2}\pi_1 = 0 \Rightarrow \frac{1}{2}\pi_1 = 0 \Rightarrow \boxed{\pi_1 = 0}$ & $\boxed{\pi_2 = 1}$

$\frac{1}{2}\pi_1 + \pi_2 = \pi_2$

$\therefore \pi = (0 \ 1)$

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(12) Define n step transition probability in a Markov chain.

Ans:

The conditional probability that the process is state a_j at step n, given that it was in state a_i at step 0. (ie) $P\{X_n=a_j / X_0=a_i\}$ is called the n-step transition probability & denoted by $P_{ij}^{(n)}$.

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(13) The random process $X(t)$ is defined as $X(t)=e^{-B|t|}$ where B is a rv uniformly distributed over $[0,2]$. Is $X(t)$ a first order stationary?

Ans: Given $X(t)=e^{-B|t|}$

$$f(B)=\frac{1}{2}, \quad 0<B<2$$

$$E[X(t)] = \int_0^2 x(t) f(B)\,dB = \int_0^2 e^{-B|t|}\frac{1}{2}\,dB$$

$$= \frac{1}{2}\left[\frac{e^{-B|t|}}{-|t|}\right]_0^2 = \frac{1}{-2|t|}\left[e^{-2|t|}-1\right] = \frac{1-e^{-2|t|}}{2|t|} \neq \text{constant}$$

$\therefore X(t)$ is not first order stationary.

(14) Let $\{X_n : n \geq 0\}$ be a Markov chain having state space $S = \{1, 2\}$ & one-step tpm given by $P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$. Find the stationary probabilities of the Markov chain.

<u>Ans:</u> Given $P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$

WKT $\pi P = \pi$ & $\pi_1 + \pi_2 = 1$ where $\pi = (\pi_1 \ \ \pi_2)$

$$(\pi_1 \ \ \pi_2) \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1 \ \ \pi_2)$$

$$\pi_1 + \frac{1}{2}\pi_2 = \pi_1$$

$$\frac{1}{2}\pi_2 = \pi_2 \Rightarrow \pi_2 - \tfrac{1}{2}\pi_2 = 0 \Rightarrow \frac{1}{2}\pi_2 = 0 \Rightarrow \boxed{\pi_2 = 0} \ \& \ \boxed{\pi_1 = 1}$$

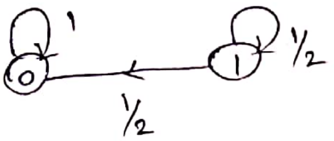
$$\therefore \pi = (1 \ \ 0)$$

---

(15) Consider a Markov chain with state space $\{0, 1\}$ & the tpm $P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$. Show that state 0 is recurrent.

<u>Ans:</u> Given

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$



<u>State 0:</u>

$$F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$$

$$f_{00}^{(1)} = 1, \quad f_{00}^{(2)} = 0, \quad f_{00}^{(3)} = 0, \ \ldots$$

$$F_{00} = 1 + 0 + 0 + \cdots = 1$$

$\therefore$ State 0 is recurrent.

QUEUEING MODELS

① In an M/M/1/∞/FCFS queue, the service rate $\mu = 1/3$ / minute & waiting time in the queue $W_q = 3$ minute, compute the arrival rate, λ.

Ans: Given $\mu = \frac{1}{3}$ per minute

$$W_q = \frac{L_q}{\lambda} = 3 = \frac{\rho L_s}{\lambda} = \frac{\rho \cdot \rho}{(1-\rho)\lambda} = \frac{\rho^2}{(1-\rho)\lambda}$$

$$\frac{\left(\frac{\lambda}{\mu}\right)^2}{\left(1-\frac{\lambda}{\mu}\right)\lambda} = 3 = \frac{\lambda}{\mu^2\left(\frac{\mu-\lambda}{\mu}\right)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$3\mu(\mu-\lambda) = \lambda \Rightarrow 3\frac{1}{3}\left(\frac{1}{3}-\lambda\right) = \lambda \Rightarrow \frac{1}{3} - \lambda = \lambda \Rightarrow 2\lambda = 1/3 \Rightarrow \boxed{\lambda = 1/6}$$

② For a M/M/c/N/FCFS ($c<N$) queueing system, write the expressions for P_0 & P_N.

Ans:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{K} \rho^{n-s}\right]^{-1}$$

$$P_n = \frac{1}{s!}\frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad s \le n \le K \quad (\text{Here } c=s \text{ \& } N=K)$$

③ In an (M/M/1/∞): FCFS queueing system, the arrival rate is $\lambda = 3$ customers/minute & utilization ratio $\rho = 0.5$. Find L_s & W_s.

Ans: Given $\lambda = 3$

$\rho = 0.5$

$$L_s = \frac{\rho}{1-\rho} = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{3}$$

④ What are the characteristics of a queueing system?

Ans: ① Arrival pattern ② Service pattern
③ Number of servers ④ System capacity
⑤ Queue discipline.

⑤ State the relationship between the average number of customers in the queue & in the system.

Ans: Average no/. of customers in the system $L_s = \frac{\rho}{1-\rho}$

Average no/. of customers in the system $L_q = \rho L_s$

⑥ Define transient & steady states.

Ans: A queueing system is in transient state when its operating characteristics are dependent on time. It is in steady state when the characteristics are independent of time.

⑦ Define balking.

Ans: Customers do not join the queue either by seeing the no/. of customers already in service system or by estimating the excessive waiting time for the desired service.

⑧ Define reneging.

Ans: Customers, after joining the queue, wait for sometime in the queue but leave before being served on account of certain reasons.

⑨ Define jockeying.

Ans: Customers move from one queue to another hoping to receive service more quickly. (a common scene at a railway booking window).

⑩ Define Kendall's notation.

Ans: Generally a queueing process is specified in symbolic form as (a|b|c):(d|e).

a - Arrival distribution
b - Service time distribution
c - No/. of servers
d - Capacity of the system
e - Queue discipline

(11) Write the types of queue discipline.

Ans:

FIFO – First in first out

LIFO – Last in first out

SIRO – Selection in random order.

(12) Explain the term traffic intensity.

Ans: $$\rho = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

(13) Define Little's formula.

Ans: $$L_s = \frac{\rho}{1-\rho}$$

$$L_q = \rho L_s$$

$$W_s = \frac{L_s}{\lambda}$$

$$W_q = \frac{L_q}{\lambda}$$

(14) Define Markovian queueing models.

Ans: Queueing models in which both inter-arrival time & service time which are exponentially distributed are called Markovian queueing models.

(15) In a given $(M/M/1):(\infty/FIFO)$ queue, $\rho = 0.6$. What is the probability that the queue contains 5 or more customers?

Ans: $P(N \geq 5) = \rho^5 = (0.6)^5$

$= 0.0778$

ADVANCED QUEUEING MODELS

① In an M/D/1 queueing system, an arrival rate of customers is 1/6 per minute & the server takes exactly 4 minutes to serve a customer. Calculate the mean number of customers in the system.

Ans: $\lambda = \frac{1}{6}$, $\mu = \frac{1}{4}$, $\rho = \frac{\lambda}{\mu} = \frac{1/6}{1/4} = \frac{4}{6} = 2/3$

$$L_s = \rho + \frac{\rho^2}{2(1-\rho)} = \frac{2}{3} + \frac{(2/3)^2}{2(1-2/3)} = \frac{2}{3} + \frac{4}{9 \times 2 \times 1/3}$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

---

② In (M/G/1) queue model, find the average no. of customer in the system if $\lambda = 1/15$ per minute, $\mu = \frac{1}{12}$ per minute, $Var(T) = 9$ minutes.

Ans: $\lambda = \frac{1}{15}$, $\mu = \frac{1}{12}$, $\sigma^2 = 9$, $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/12} = \frac{12}{15} = \frac{4}{5}$

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{(1/15)^2(9) + (4/5)^2}{2(1-4/5)} + \frac{4}{5} = \frac{0.04 + 0.64}{0.4} + \frac{4}{5} = \frac{0.68}{0.4} + \frac{4}{5}$$

$$\therefore L_s = 1.7 + \frac{4}{5} = 2.5$$

---

③ Define series queue model.

Ans: A series queue model or a tandem queue model is satisfies the following characteristics.

(i) Customers may arrive from outside the system at any node & may leave the system from any node.

(ii) Customers may enter the system at some node, traverse from node to node in the system & leave the system from same node, necessarily following the same order of nodes.

(iii) Customers may return to the nodes already visited, skip some nodes & even choose to remain in the system forever.

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④ In an M/D/1 : FCFS queueing system, an arrival rate of customers is 10 per second & a service rate of customers is 20 per second. Compute the mean no/- of customers in the system.

Ans: $\lambda = 10$, $\mu = 20$, $\sigma^2 = 0$, $\rho = \frac{\lambda}{\mu} = \frac{10}{20} = \frac{1}{2}$

$$L_s = \rho + \frac{\rho^2}{2(1-\rho)} = \frac{1}{2} + \frac{\frac{1}{4}}{2(1-\frac{1}{2})} = \frac{1}{2} + \frac{2}{8} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

---

⑤ Consider a two-station tandem Markovian queueing network with customers arrival rate of $\lambda = 2$/min/. & service rates $\mu_1 = 4$/min/. at station-1 & $\mu_2 = 6$/min/. at station-2. Compute the waiting time of a customer in the system & the probability that both the servers are idle.

Ans: $\lambda = 2$, $\mu_1 = 4$, $\mu_2 = 6$

$$W_s = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{4-2} + \frac{1}{6-2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$P(0,0) = \left(\frac{\lambda}{\mu_1}\right)^0 \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^0\left(1 - \frac{\lambda}{\mu_2}\right)$$

$$= (1)\left(1 - \frac{2}{4}\right)(1)\left(1 - \frac{2}{6}\right) = \frac{2}{4} \times \frac{4}{6} = \frac{1}{3}$$

---

⑥ What do the letter in the symbolic representation M/G/1 of a queueing model represent?

Ans: M → Inter arrival time is exponential distribution

G → Service time is general distribution.

1 → No/. of server.

---

⑦ Consider a tandem queue with 2 independent Markovian servers. The situation at server 1 is just as in an M/M/1 model. What will be the type of queue in server 2? Why?

Ans: The type of queue in server 2 is also a M/M/1 model. Since output of M/M/1 is another M/M/1 queue.

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(8) When a M/G/1 queueing model will become a classic M/M/1 queueing model?

<u>Ans:</u> In the M/G/1 model, G stands for the general service time distribution. If G is replaced by exponential service time distribution then the M/G/1 model becomes the classical M/M/1 model.

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(9) Define a tandem queue.

<u>Ans:</u> A series queue in which the service facilities are arranged in sequence & the flow is always in a single direction.

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(10) Give any 2 examples for series queueing situations.

<u>Ans:</u>

(1) Manufacturing or assembly line process.

(2) Clinic physical examination procedure.

(3) Registration process in university.

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(11) Write down the (flow balance) traffic equations for an open Jackson network & stability condition of the system.

<u>Ans:</u> Jackson's flow balance equations for this open model are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$, $j = 1, 2, \ldots, k$.

Stability condition: $\frac{\lambda_j}{\mu_j} < 1$, $j = 1, 2, \ldots, k$.

$\lambda_j \rightarrow$ arrival rate, $\mu_j \rightarrow$ service rate.

---

(12) What do you mean by series queue with blocking?

<u>Ans:</u> This is a sequential queue model consisting of two service points S_1 & S_2, at each of which there is only one server & where no queue is allowed to form at either point.

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(13) What do you mean by bottleneck of a network.

Ans: The service station for which the utilization factor is maximum among all the other service stations of the network is called the bottleneck of a network.

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(14) Define open queueing network.

Ans: An open queueing network is characterised by one or more sources of job arrivals & corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes then it must be open for each class of jobs.

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(15) Define closed queueing network.

Ans: In a closed queueing network, jobs neither enter nor depart from the network. If the network has multiple job classes then it must be closed for each class of jobs.
